# Mini-courses

## Contemporary theoretical concepts of dynamical chaos

*Sergey Gonchenko, Nizhny Novgorod State University, Russia.*

Abstract: The presentation consists of three lectures and two exercise sessions.
Lecture 1. Three types of dynamical chaos.
Lecture 2. Elements of modern theory of strange attractors.
Lecture 3. Variety of pseudohyperbolic strange attractors in three-dimensional Hénon-like maps.
ES1. Example of systems with reversible mixed dynamics, computer experiments (together with A. Kazakov).
ES2. Computer and qualitative methods of search of strange attractors in three-dimensional Hénon-like maps (together with A. Gonchenko).

In Lecture 1 we discuss new concept of dynamical chaos based on the existence of three independent form of chaos: conservative chaos, strange attractor and mixed dynamics. Two first forms of chaos are well-known, while the mixed dynamics is rather new. Formally, such classification appears by the following principle: conservative chaos – attractor and repeller coincide; strange attractor (repeller) – attractor and repeller do not intersect; mixed dynamics – attractor and repeller intersect but do not coincide. We give mathematical concept for all three forms of chaos based on the Anosov-Conley-Ruelle theory of chaos and notion of $\varepsilon$-orbit. Main attention will be paid to the mathematical concept of mixed dynamics.

In Lecture 2 we present some elements of modern theory of strange attractors. We divide all attractors into two classes: quasiattractors and pseudohyperbolic attractors. In particular, pseudohyperbolic attractors include hyperbolic and Lorenz attractors but can contain also the so-called wild hyperbolic attractors, which allow homoclinic tangencies and, hence, Newhouse wild hyperbolic sets, but bifurcations of these homoclinic tangencies do not lead to the birth of periodic sinks (quasiattractors, conversely, allow the birth of periodic sinks, by definition). The theory of wild hyperbolic and pseudohyperbolic attractors was laid by Turaev and Shilnikov [1,2] and we discuss some elements of this theory and give several examples of such attractors.

In Lecture 3 we discuss the problem on the existence of pseudohyperbolic attractors of various types in three-dimensional generalized Hénon maps. We recall, first of all, results of [3] on existence of discrete Lorenz attractors in 3d Hénon maps. Next, we present new types of pseudohyperbolic attractors, in particular, various types of discrete figure-8 attractors.

During ES1 we present some mathematical models demonstrating reversible mixed dynamics, including models of dynamics of rigid body.

During ES2 we discuss and demonstrate in action some very useful computer tools of searching strange attractors on example of 3d generalized Hénon maps.


[1] D.V.Turaev and L.P.Shilnikov, An example of a wild strange attractor.- Sb. Math., 1998., v.189, No.2, 137–160.

[2] D.V. Turaev, L.P. Shilnikov, Pseudohyperbolicity and the problem on periodic perturbations of Lorenz-like attractors.- Russian Dokl. Math., 2008, v.467, 23–27.

[3] S.V. Gonchenko, I.I. Ovsyannikov, C. Simo, D. Turaev, Three-dimensional Henon-like maps and wild Lorenz-like attractors.- Int.J. Bifurcation and Chaos, 2005, v.15, 3493-3508.


# Fractal attractors in skew product systems


*Tobias Jäger, Friedrich-Schiller-Universität Jena, Germany.*



Abstract: The course gives and introduction to the study of fractal properties of attractors and repellers in skew product systems, including the case of the famous Weierstrass graphs.

The lectures thereby falls into two parts: the first part of the course will provide detailed proofs of more elementary facts concerning Hölder regularity and box dimension of attracting invariant graphs in skew products with contracting fibres, aiming to give a solid intuition and insight into the methods of the field. In the second part, we will cover slightly more advanced topics, and in particular discuss how Ledrappier-Young theory and results of Tsujii and Shen on fat solenoidal attractors have been applied by Baranski, Barany and Romanowska to determine the Hausdorff dimension of Weierstrass graphs.

## Symbolic dynamics for hyperbolic surfaces with cusps, and applications to Laplace eigenfunctions

*Anke Pohl, Max Planck Institue for Mathematics Bonn, Germany.*

Abstract: For the investigation of topological or smooth dynamical systems it is sometimes possible and very fruitful to model (usually lossy) the dynamical system by a discrete version, a so-called symbolic dynamics, which captures the essentials of the properties one is interested in. We will discuss a construction of symbolic dynamics for the geodesic flow on hyperbolic surfaces with cusps, and will show how they can be applied in the study of Laplace eigenfunctions.

## Dynamics of non-hyperbolic iterated function systems

*Dmitry Turaev, Imperial College London, UK.*

# Public lecture

## The Ubiquity of Dynamical Systems

*Anke Pohl, Max Planck Institue for Mathematics Bonn, Germany.*

Abstract: Suppose you have a huge space, say the earth or a billiard table, and a very small marble sitting on this space. Now you push the marble and observe how it travels across the space, bounces off walls and so on. Some typical questions you might have are:

- Will its trajectory come very close to every point on the space? Or will it be periodic? Or will it be trapped in some part of the space?
- If I launch a second marble near the first one, will their trajectories stay close to each other for a long time?

Such dynamical systems and the answers to these questions have many applications not only in mathematics and physics but also for surprisingly many daily-life problems such as, e.g., designing an opera hall or detecting accounting fraud. In this lecture we will provide an easy accessible introduction to this fascinating area and some of its current research problems.

# Plenary talks

## Multi-Chaos and Quasiperiodicity

*Jim Yorke, University of Maryland, USA.*

Abstract: Chaos and quasiperiodicity are the two complicated nonlinear dynamical behaviors seen in typical systems and this talk is about both. The cat map on the 2-torus has a dense set of saddle periodic points. The map $(x, y) \to (2x \mod 1, 3x \mod 1)$ on the 2-torus has a dense set of repelling periodic points. I will describe a smooth map on the 2-torus that has "multi-chaos" in the sense that the 2-torus has both a dense set of saddle periodic points and a dense set of repelling periodic points. Does that sound impossible? We believe we have the first rigorous example of such a map. Our example is rather robust under perturbations. And we expect such behavior to be very common or predominant in systems with high-dimensional chaotic attractors. Our map also has a quasiperiodic orbit. The problem of distinguishing between chaotic points and quasiperiodic points when both exist in the same system brings us to the Birkhoff Ergodic Theorem. Given a trajectory $(xn)$ and a real-valued function $f$, let $BN(f)$ be the average of $\{f(xn)\}$ where $n = 0, \ldots, N$. The theorem concludes that the average $BN(f)$ converges to the space average as $N \to \infty$. The space average is an integral that we often want to compute. But convergence is so Extremely Slow that it is practically useless as a computational tool. I will show how for $C^\infty$ quasiperiodic processes with a $C^\infty$ $f$, by a small modification of $BN$ we speed up convergence by a factor of 1025 when requiring 30-digit accuracy. That is, if our method converges in one second, the Birkhoff average would take about a billion billion years. Our method has no speed up for chaotic systems. See Measuring Quasiperiodicity: http://arxiv.org/abs/1512.07286

## Deformation of Transfer Operators

*Alexander Adam, Université Pierre et Marie Curie, France.*

Abstract: For any cofinite Hecke triangle surface it is well-known that the odd $L^2$-spectrum of the Laplacian satisfies the Weyl law. In stark contrast, the Phillips-Sarnak conjecture states that for generic Hecke triangle surfaces the even spectrum should be finite. Some supporting results have been obtained by Judge and Hillairet-Judge, which lead to the even stronger conjecture that the even spectrum should be trivial. The results by Judge and Hillairet-Judge as well as the heuristics of the Phillips-Sarnak conjecture rely

on a deformation theory along a certain one-parameter family of hyperbolic triangles. I will report on the current status of our attempt to establish such a deformation theory with transfer operator techniques.

# Dynamics of Discrete Time Systems with the Hysteresis Stop Operator

*Nikita Begun, Free University Berlin, Germany.*

Abstract: We consider a piecewise linear two-dimensional dynamical system that couples a linear equation with the so-called stop operator. Global dynamics and bifurcations of this system are studied depending on two parameters. The system is motivated by general-equilibrium macroeconomic models with sticky information.

# On quadritic homoclinic tangencies in 2D symplectic maps

*Marina Gonchenko, University of Uppsala, Sweden.*

Abstract: I will overview the results related to the study of orbit behaviour near a nontransversal homoclinic orbit. Let $f_0$ be a two-dimensional symplectic diffeomorphism with a saddle fixed point, whose stable and unstable invariant manifolds have a quadratic homoclinic tangency. Consider also a family of two-dimensional symplectic $C^r$-maps $f_\varepsilon$ close to $f_0$. Our aim is to study bifurcations of single-round periodic points in family $f_\varepsilon$. Every point of such an orbit can be considered as a fixed point of the corresponding *first return map* that is constructed as a superposition $T_k = T_1 T_0^k$ of two maps $T_0 \equiv T_0(\varepsilon)$ and $T_1 \equiv T_1(\varepsilon)$. The map $T_0$ is called *local map* and is defined as the restriction of $f_\varepsilon$ onto a neighbourhood $U_0$ of the saddle point, while the map $T_1$ is called *global map* and is defined as $T_1 \equiv f_\varepsilon^q$ and acts through small disks surrounding those points of the homoclinic orbit that do not lie in $U_0$. We use the so-called finitely smooth normal forms for the local map and construct the map $T_k$. Studying bifurcations of fixed points of $T_k$, we prove the existence of cascades of elliptic periodic points for one and two parameter general unfoldings $f_\varepsilon$.

# Random interval maps

*Ale Jan Homburg, University of Amsterdam, Netherlands.*

Abstract: We discuss the dynamics of skew product systems that correspond to iterated function systems of interval maps.

# Weierstrass meets negative Schwarzian

*Gerhard Keller, Universität Erlangen-Nürnberg, Germany.*

Abstract: The classical Weierstrass graph and its generalizations can be interpreted as the attractor of skew product dynamical systems with affine one-dimensional branches driven by (the stable coordinate of) a baker transformation. New effects occur, when the affine branches are replaced by sigmoidal branches with negative Schwarzian derivative. I present some results highlighting the role that strong stable fibres play also in this setting, and I will discuss some numerical observations showing transient behaviour and metastability with exceedingly long transition times.

# Combinatorial constructions in Smooth Ergodic Theory

*Phillip Kunde, Universität Hamburg, Germany.*

Abstract: Until 1970 it was an open question if there is an ergodic area-preserving smooth diffeomorphism on the disc $\mathbb{D}^2$. This problem was solved by the so-called "approximation by conjugation"-method developed by D. Anosov and A. Katok. In fact, on every smooth compact connected manifold of dimension $m \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{S}^1}$ preserving a smooth volume $\nu$ this method enables the construction of smooth diffeomorphisms with particular ergodic properties or non-standard smooth realizations of measure-preserving systems. In my talk, I will present a new construction of weakly mixing diffeomorphisms preserving a measurable Riemannian metric in the restricted space
$\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} \ : h \in \mathrm{Diff}^\infty(M,\nu)\}}^{C^\infty}$ for a given Liouvillean number $\alpha \in \mathbb{S}^1$. So we design maps with prescribed rotation number.

# Symmetric bifurcation with high dimensional kernels

*Reiner Lauterbach, Universität Hamburg, Germany.*

Abstract: We present a systematic way to generate equivariant bifurcation problems from equilibria where the kernel of the linearisation is even dimensional. In particular we study cases where this dimension is four and were all isotropy subgroups have even dimensional fixed point spaces. In all cases which we have looked at we found nontrivial isotropy types which are generically symmetry breaking. At the same time we have indications of rather complicated dynamics on those center subspaces.

# Open dynamical systems with infinite invariant measures

*Sara Munday, University of Bologna, Italy.*

Abstract: We investigate the scaling of the escape rate from piecewise-linear dynamical systems displaying intermittency due to the presence of an indifferent fixed-point. Strong intermittent behaviour in the dynamics can result in the system preserving an infinite measure. We define a neighbourhood of the indifferent fixed point to be a hole through which points escape and investigate the scaling of the rate of this escape as the length of the hole decreases, both in the finite measure preserving case and infinite measure preserving case. In the infinite measure preserving systems we observe logarithmic corrections to and polynomial scaling of the escape rate with hole length. Finally we conjecture a relationship between the wandering rate and the observed scaling of the escape rate.

# Volume hyperbolicity and wildness

*Katsutoshi Shinohara, Hitotsubashi University, Japan.*

Abstract: It is known that the robust absence of volume hyperbolicity (a weaker notion of hyperbolicity) on a chain recurrence class implies the generic coexistence of inifintely many sinks or sources, in particular the wildness (generic coexistence of infinitely many distinct chain recurrence classes) of the system. In other words, volume hyperbolicity is a necessary condition for the tameness (robust finitude of number of chain recurrence classes). In this talk, I will show that volume hyperbolicity is not a sufficient condition for the tameness, by giving examples of diffeomorphisms on any three manifold which is volume hyperbolic and simultaneously wild. As a by-product, I give an example of open region of diffeomorphisms in which generic diffeomorphism has neither topological attractors nor repellers. The construction comprises the study of the shape of the attracting region and the repelling region from the (differential) topological viewpoint and the $C^1$ perturbation technique called flexible periodic points.

# Shadowing of finite pseudo-orbits

*Sergey Tikhomirov, Max Planck Institue for Mathematics Leipzig, Germany.*

Abstract: We consider dynamical systems generated by diffeomorphisms. It is well-known that for uniformly hyperbolic diffeomorphisms pseudo-orbits can be shadowed by true trajectories for infinite time. Hammel-Grebogi-Yorke conjectured that for a wide class of non-uniformly hyperbolic diffeomorphisms d-pseudo-orbits of length $1/\sqrt{d}$ can be $\sqrt{d}$-shadowed. We proved that this conjecture cannot be improved and for a special case of non-uniformly hyperbolic diffeomorphisms found precise length of shadowable pseudo-orbits. The main techniques are finite difference equations and large deviation principle.

# Spanning Trees, Sandpiles, and Algebraic Dynamics

*Evgeny Verbitskiy, Leiden University, Netherlands.*

Abstract: In 1993, Burton and Pemantle posed a question about the relation between uniform spanning forests on lattices $Z^d$ and algebraic dynamical systems with equal entropy. It has been conjectured that uniform spanning forests form symbolic covers of the corresponding algebraic systems. In the talk, I will review some partial progress in understanding this intriguing relation and will discuss a new example: uniform spanning trees on ladder graphs. The talk is based on joint works with K. Schmidt (Vienna) and T. Shirai (Fukuoka).

# Dimensions of limit sets of Kleinian groups

*Kurt Falk, Universität Bremen, Germany.*

Abstract: The dynamics of geometrically finite hyperbolic manifolds, where recurrence and ergodicity play a central role, is well understood by means of Patterson-Sullivan theory. For geometrically infinite manifolds or manifolds given by infinitely generated Kleinian groups, non-recurrent dynamics becomes the ”thick part” of dynamics, not only in measure but often also in (Hausdorff) dimension. I will discuss Hausdorff and Minkowski dimension of limit sets of Kleinian groups and the connection between these notions and dynamics within the convex core of the associated hyperbolic manifold.